Lecture - 39.

Gauss's Divergence Theorem.

Let D be a closed and bounded region in 3-Dim. space whose boundary is a piecewise smooth surface S that is oriented outwards.

Let $\vec{V} = (V_1\hat{i} + V_2\hat{j} + V_3\hat{k})$ be a vector field which is continuous and has continuous first order partial derivatives in some domain containing D.

Let $\hat{n}$ is the unit normal vector to S drawn outwards.

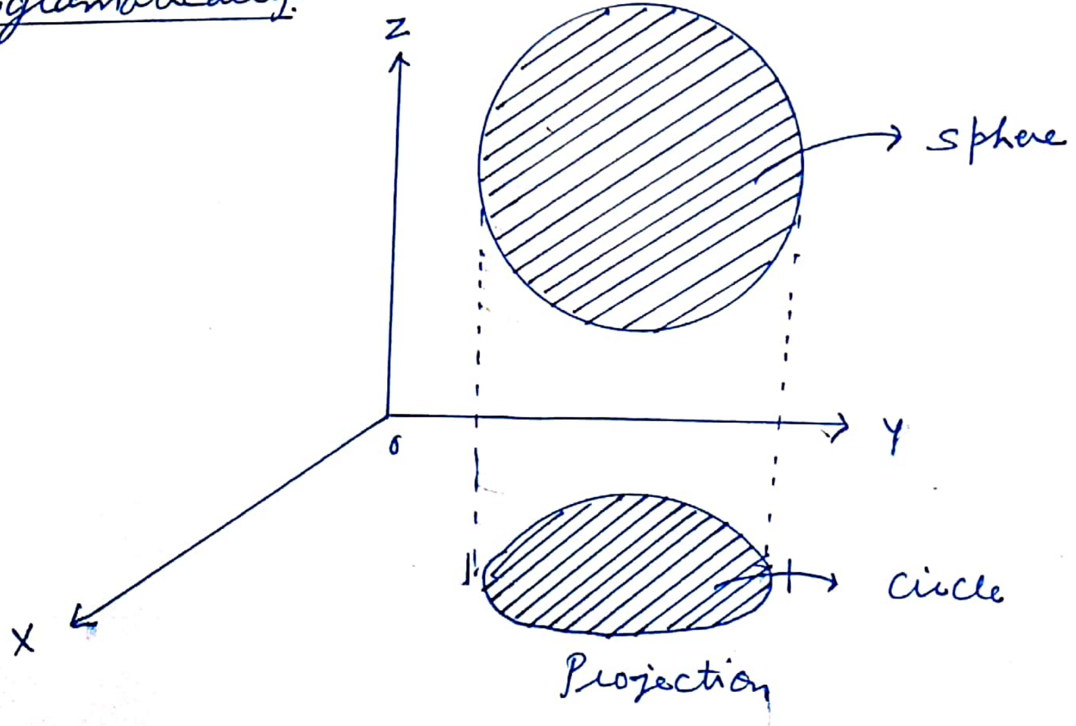
Then:

$$\boxed{\iint_S (\vec{V}\cdot\hat{n})\,dA = \iiint_D (\vec{\nabla}\cdot\vec{V})\,dv}$$

or

$$* \quad \boxed{\iint_S (\vec{V}\cdot\hat{n})\,dA = \iiint_D [\operatorname{div}(\vec{V})]\,dv}$$

· Diagramatically:



Evaluate $\iint_S (\vec{V}\cdot\hat{n})\,dA$, Using Gauss's div. thm.

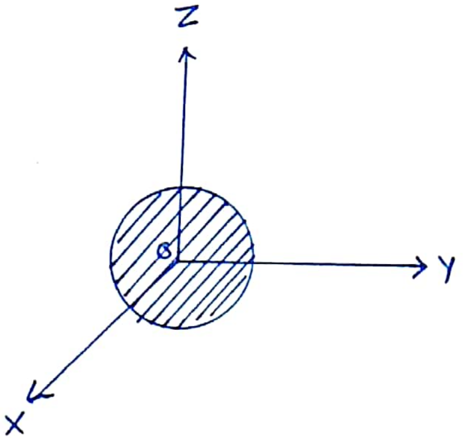
(1). $\vec{V} = (x\hat{i} + y\hat{j} + z\hat{k})$, D: Region bounded by the sphere $x^2+y^2+z^2=16$.

Sol: $\vec{V} = (x\hat{i} + y\hat{j} + z\hat{k})$

$$\Rightarrow \operatorname{div}(\vec{V}) = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 1 + 1 + 1$$

$$\Rightarrow \boxed{\operatorname{div}(\vec{V}) = 3}$$



By Gauss div. Thm.

$$\iint_S (\vec{V}\cdot\hat{n})\,dA = \iiint_D [\operatorname{div}(\vec{V})]\,dv$$

$$= \iiint_D (3)\,dv$$

$$= 3\iiint_D dv$$

$$= 3\,(\text{volume of sphere } x^2+y^2+z^2=16)$$

$$= \cancel{3}\left(\frac{4}{\cancel{3}}\pi r^3\right)$$

$$= 4\pi(4)^3$$

$$= 256\pi \quad \text{Ans.}$$

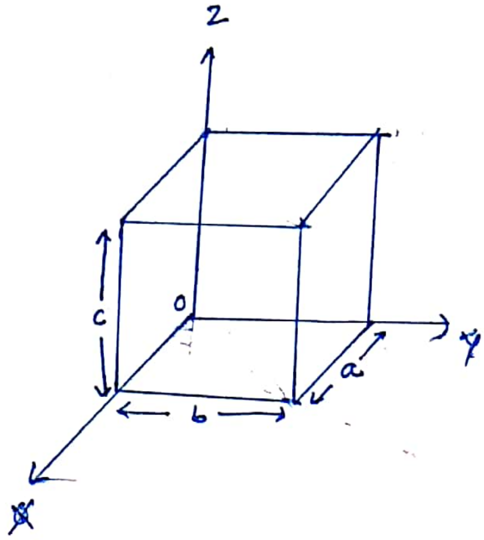
(2). $\vec{V} = (x\hat{i} + y\hat{j} + z\hat{k})$, D: D is bounded by region $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$.

Sol: $\vec{V} = (x\hat{i} + y\hat{j} + z\hat{k})$

$$\Rightarrow \text{div}\,\vec{V} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 1 + 1 + 1 = 3$$

$$\Rightarrow \boxed{\text{div}(\vec{V}) = 3}$$



By Gauss's Div Thm:

$$\iint_S (\vec{V}\cdot\hat{n})\,dA = \iiint_D (\text{div}\,\vec{V})\,dv$$

$$= \iiint_D (3)\,dv$$

$$= 3\iiint_D dv$$

$$= 3\left[(\text{Volume of cuboid});\ \begin{matrix} 0 \le x \le a \\ 0 \le y \le b \\ 0 \le z \le c \end{matrix}\right]$$

$$= 3(lbh)$$

$$= 3abc \qquad \text{Ans}$$

Imp. Result:
If $\vec{r} = (x\hat{i} + y\hat{j} + z\hat{k})$; D is a bounded region with volume v, then

$$\boxed{\iint_S (\vec{r}\cdot\hat{n})\,dA = 3v = 3(\text{Volume})}$$